FECHNERIAN SCALING AND THURSTONIAN MODELING

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Abstract

Fechnerian scaling, in the Dzhafarov-Colonius version, uses "same-different" discrimination probability functions $\psi(x, y)$ to define on stimulus spaces certain (Fechnerian) metrics. A Thurstonian representation for $\psi(x, y)$ is understood as a model in which stimuli x, y are mapped into random variables P, Q whose density functions are defined on a hypothetical perceptual space, the discrimination judgment in every trial being determined by their realizations in this trial. The paper describes the implications for both Fechnerian scaling and Thurstonian modeling of the following two properties of $\psi(x, y)$: (1) if $y = b$ is the minimum of function $y \mapsto \psi(a, y)$ (the point of subjective equality, PSE, for $x = a$), then $x = a$ is the minimum of function $x \mapsto \psi(x, b)$ (the PSE for $y = b$); (2) the minimum level $\psi(a, b)$ of $\psi(x, y)$ is generally different for different PSE-pairs a, b. It turns out, in particular, that no "well-behaved" Thurstonian model (e.g., with multivariate normal densities smoothly depending on stimuli) can account for such $\psi(x, y)$.

1 Two basic properties of discrimination

When two stimuli x, y are presented for a discrimination judgment, they necessarily belong to distinct *observation intervals*, OI_1 and OI_2 (time intervals and/or spatial locations). While classical psychophysics is primarily concerned with "greater-less" discriminations defined with respect to semantically unidimensional attributes, such as brightness or loudness, this paper is focused on generic, "same-different" discriminations: $x \approx y$ or $x \not\approx y$ (ignoring the difference in OI). Denoting $\psi(x,y) = \Pr[x \not\approx y]$, functions $y \mapsto \psi(x,y)$ and $x \mapsto \psi(x,y)$ achieve their minima at certain points $y = h(x)$ and $x = g(y)$, respectively, with $\psi(x, h(x) + s)$ and $\psi(g(y) + s, y)$ increasing in $|s|$. Stimulus $h(x)$ can be called the *point of subjective equality* (PSE) in OI_2 for x taken in OI_1; $g(y)$ is the PSE in OI_1 for y taken in OI_2.

The discrimination probability function $\psi(x,y)$ is said to possess the *regular minimality* property if $g \equiv h^{-1}$. With this property posited, $\psi(x,y)$ can always (and is hereafter assumed to) be presented in a *canonical form*, with $h(a) = g(a) = a$, so that $y \mapsto \psi(x,y)$ and $x \mapsto \psi(x,y)$ achieve their minima at $x = y$.

Regular minimality is corroborated by empirical data. It is also known from data that the minimum level $\psi(x,x)$ of $\psi(x,y)$ generally varies with x. This fact is referred to as the *nonconstant self-similarity* property.

The analysis below applies to stimuli varying in several continuous physical characteristics, but to save space, the discussion will be confined to unidimensional stimuli only, $x, y \in (a_{\inf}, a_{\sup})$. The general theory is presented in Dzhafarov & Colonius (1999, 2001), Dzhafarov (in press, a, b), and (as of August 2001) in Dzhafarov (submitted, a, b, c).

2 Thurstonian analysis

Consider a Thurstonian model is which x, y are mapped into random variables $\mathbf{P}, \mathbf{Q}$ taking on their values in some perceptual space $\mathfrak{P} \subseteq \mathrm{Re}^m$ and having density functions $\alpha(\mathbf{p}, x), \beta(\mathbf{q}, y)$. For simplicity, assume that $\mathbf{P}, \mathbf{Q}$ are stochastically independent and that the decision rule is deterministic, i.e., $\mathfrak{P} \times \mathfrak{P}$ is partitioned into $\mathfrak{S}$ and $\overline{\mathfrak{S}}$, such that "$x \not\approx y$" iff $(\mathbf{p}, \mathbf{q}) \in \mathfrak{S}$ and

$$\psi(\mathbf{x}, \mathbf{y}) = \Pr[(\mathbf{P}, \mathbf{Q}) \in \mathfrak{S}] = \int_{(\mathbf{p},\mathbf{q}) \in \mathfrak{S}} \alpha(\mathbf{p}, \mathbf{x}) \beta(\mathbf{q}, \mathbf{y}) \, d\mathbf{q} d\mathbf{p}.$$

Can one choose $\mathfrak{P}, \alpha, \beta, \mathfrak{S}$ so that $\psi(x,y)$ possesses the regular minimality and nonconstant self-similarity properties?

The answer is affirmative if one imposes no restrictions on α, β and on $\mathfrak{S}$, allowing, in particular, α and β to be delta functions: then, a theorem says, any $\psi(x,y)$ whatsoever has a Thurstonian representation with independent $\mathbf{P}, \mathbf{Q}$ and a deterministic decision rule.

The answer turns out to be negative, however, if one considers only "*well-behaved*" Thurstonian models. The well-behavedness means that (1) α, β are bounded and piecewise continuous, and their continuity bounds and values within these bounds change in response to stimulus changes "sufficiently smoothly" and "not too fast" (which translates into the existence of left and right derivatives $\frac{\partial}{\partial x\pm}$, $\frac{\partial}{\partial y\pm}$ dominated by appropriately integrable functions); (2) the decision rule is "reasonable", i.e., $\mathfrak{S}$ consists of several areas with continuous boundaries.

Thus, any Thurstonian model will be well-behaved if α, β in it are constructed from "textbook" densities (multivariate normal, gamma, uniform, etc.) with parameters sufficiently smoothly depending on stimuli, and if the decision rule is, say, category-based ($x \approx y$ iff $\mathbf{p}$ and $\mathbf{q}$ belong to the same element of some partitioning of $\mathfrak{P}$) or continuous distance-based ($x \not\approx y$ iff $\mathbf{p}$ and $\mathbf{q}$ are farther than ε apart). A theorem says that *no such model can account for* $\psi(x, y)$ *subject to regular minimality and nonconstant self-similarity.*

The same conclusion is reached when the notion of well-behavedness is extended to Thurstonian models with *probabilistic decision rules*, where each $(\mathbf{p}, \mathbf{q})$ leads to the judgment "$x \not\approx y$" with some probability $P(\mathbf{p}, \mathbf{q})$, and to Thurstonian models with *stochastically interdependent* $\mathbf{P}, \mathbf{Q}$, provided $\mathbf{P}$ and $\mathbf{Q}$ can be *selectively attributed* to x and y, respectively (see Dzhafarov, 2001, for a discussion of selective attribution under stochastic interdependence).

The reason for this failure of well-behaved Thurstonian models is that they generate $\psi(x, y)$ with the "*near-smoothness*" property: $\frac{\partial}{\partial x\pm}\psi(x, y)$ exist and are continuous in y, and $\frac{\partial}{\partial y\pm}\psi(x, y)$ exist and are continuous in x. It can be shown, however, that regular minimality and nonconstant self-similarity are incompatible with near-smoothness. In fact, the class of Thurstonian models that predict near-smooth functions $\psi(x, y)$ and should therefore be rejected is much broader than just the class of well-behaved models.

3 Fechnerian analysis

Fechnerian scaling is based on certain assumptions about the shape of functions $y \mapsto \psi(x, y)$ and $x \mapsto \psi(x, y)$ in the vicinity of $x = y$. The main consequence of these assumptions is the asymptotic equalities (as $s \to 0+$)

$$\begin{cases} \psi(x, x \pm s) - \psi(x, x) \sim \left[F_1^{\pm}(x) R(s)\right]^{\mu} & \text{(for } x \text{ in } \mathrm{OI}_1) \\ \psi(x \pm s, x) - \psi(x, x) \sim \left[F_2^{\pm}(x) R(s)\right]^{\mu} & \text{(for } x \text{ in } \mathrm{OI}_2) \end{cases}.$$

Here, $\mu > 0$ is called the *psychometric order* of the stimulus space, $R(s)$ is a function *regularly varying at* $s = 0$ *with unit exponent* (e.g., s, $s \log \frac{1}{s}$, etc.; see Dzhafarov, in press, a), and the positive, continuous Fechner-Finsler *metric functions* $F_1^+(x)$, $F_1^-(x)$, $F_2^+(x)$, $F_2^-(x)$, generally all different, are used for

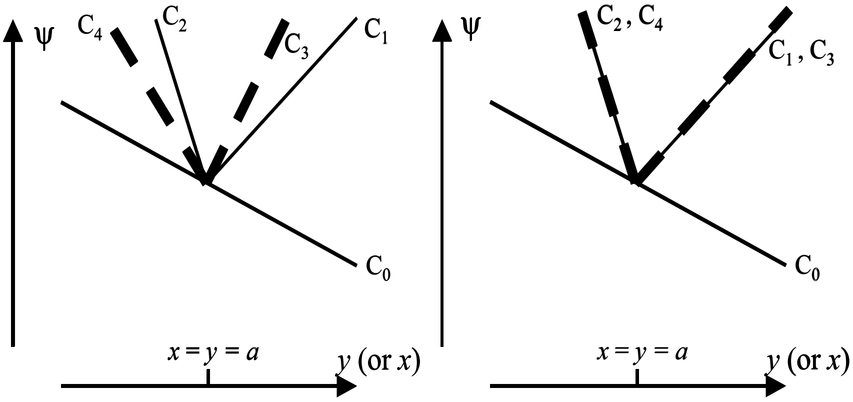

Figure 1: (Case A) Possible appearances of $\psi(a, y)$ (C_1C_2) and $\psi(x, a)$ (C_3C_4) in a very small vicinity of $x = y = a$. Tangent slopes of C_1, C_2, C_3, C_4 are $F_1^+(x), -F_1^-(x), F_2^+(x), -F_2^-(x)$. The tangent slope of $\psi(x, x)$ (C_0) is $F_1^+(x) - F_2^-(x) = F_2^+(x) - F_1^-(x)$.

computing *Fechnerian distances* (generally *oriented*, i.e., non-symmetric):

$$\begin{cases} G_1(a,b) = \int_a^b F_1^+(x)\,dx; \quad G_1(b,a) = \int_a^b F_1^-(x)\,dx \quad (\text{for } a \le b \text{ in } \mathrm{OI}_1) \\ G_2(a,b) = \int_a^b F_2^+(x)\,dx; \quad G_2(b,a) = \int_a^b F_2^-(x)\,dx \quad (\text{for } a \le b \text{ in } \mathrm{OI}_2) \end{cases}.$$

It turns out that the conjunction of regular minimality and nonconstant self-similarity is only compatible with the following two special cases of the Fechnerian theory.

(**Case A**, see Fig. 1) $\mu = 1$, $R(s) \equiv s$, and

$$\begin{array}{c} F_1^+(x) + F_1^-(x) = F_2^+(x) + F_2^-(x) \\ F_1^+(x) - F_2^-(x) = F_2^+(x) - F_1^-(x) = \lim\limits_{s\to 0} \frac{\psi(x+s,x+s)-\psi(x,x)}{s} \end{array},$$

from which it follows that, for $a \le b$,

$$\begin{array}{c} G_1(a,b) + G_1(b,a) = G_2(a,b) + G_2(b,a) \\ G_1(a,b) - G_2(b,a) = G_2(a,b) - G_1(b,a) = \psi(b,b) - \psi(a,a). \end{array}$$

Metrics G_1 and G_2 in this case cannot be symmetric: $G_1(a,b) \not\equiv G_1(b,a)$, $G_2(a,b) \not\equiv G_2(b,a)$.

(**Case B**, see Fig. 2) $\mu \le 1$, $R^\mu(s)/s \to \infty$ as $s \to 0+$ (i.e. $R(s)$ may be s if $\mu < 1$ and, say, $s \log \frac{1}{s}$ if $\mu = 1$), and

$$F_1^+(x) = F_2^-(x), \quad F_2^+(x) = F_1^-(x) \ ,$$

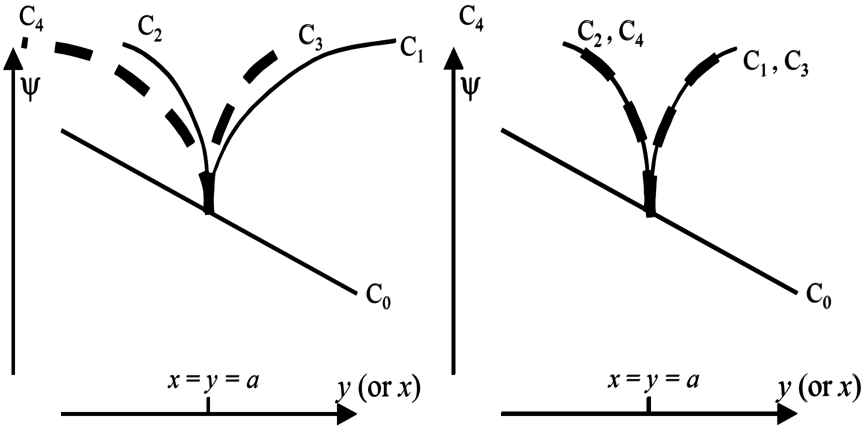

Figure 2: (Case B) Possible appearances of $\psi(a, y)$ (C_1C_2) and $\psi(x, a)$ (C_3C_4) in a very small vicinity of $x = y = a$. C_3C_4 is the mirror-reflection of C_1C_2; they coincide iff they are bilaterally symmetric.

whence

$$G_1(a, b) = G_2(b, a), \quad G_2(a, b) = G_1(b, a).$$

Metrics G_1 and G_2 in this case may or may not be symmetric.

Since in both these cases μ cannot exceed 1, we conclude that in the vicinity of their minima the functions $y \mapsto \psi(x, y)$ and $x \mapsto \psi(x, y)$ are $\vee$-shaped or $\curlyvee$-shaped (Figs. 1 and 2) rather than $\cup$-shaped. This accords with the mentioned earlier fact that under regular minimality and nonconstant self-similarity $\psi(x, y)$ is not near-smooth.

4 Conclusion

If regular minimality and nonconstant self-similarity are posited to be basic properties of $\psi(x, y)$, as they seem to be, the use of Thurstonian modeling as the sole explanatory scheme for this function should be called into question. Whatever one's view of the restrictiveness and plausibility of the well-behavedness constraints, the results presented in this paper prompt one to look for viable alternatives to the mathematical metaphor of stimuli individually mapped into random variables. The "*probability-distance hypothesis*" (Dzhafarov, in press, b) according to which $\psi(x, y)$ is a monotone function of some distance $D(x, y)$ on stimulus space cannot be such an alternative as it cannot account for nonconstant self-similarity. My expectation is, however, that Fechnerian distances should be among principal determinants of the probabilities with which stimuli are discriminated from each other. The possible forms of this determination, as well as the identity of other, "non-Fechnerian" determinants of $\psi(x, y)$, turn out to critically depend on which of the two special cases considered in Section 3, A or B, is supported by empirical evidence. The answer as yet is not known.

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